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ON THE STABILITY OF A GENERAL ADDITIVE FUNCTIONAL INEQUALITY IN BANACH SPACES

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(2x_1) + f(2x_2) + \cdots + f(2x_n)\| \leq \|tf(x_1 + x_2 + \cdots + x_n)\|$$

in Banach spaces where a positive integer $n \geq 3$ and a real number t such that $2 \leq t < n$.

1. Introduction

In 1940, S. M. Ulam [4] suggested the stability problem of functional equations concerning the stability of group homomorphisms.

In the next year, D. H. Hyers [1] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces as follows: *If $\delta > 0$ and if $f : \mathcal{X} \to \mathcal{Y}$ is a mapping between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ satisfying*

$$\left\|f(x+y) - f(x) - f(y)\right\| \leq \delta$$

for all $x, y \in \mathcal{X}$, then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that $\left\|f(x) - A(x)\right\| \leq \delta$ for all $x, y \in \mathcal{X}$.

This type is called the Hyers-Ulam stability.

Throughout this paper, let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a Banach space. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. In 2007, C. Park, Y. S. Cho and M. H. Han [3] proved the generalized Hyers-Ulam stability of the additive functional inequality

$$\left\|f(x) + f(y) + f(z)\right\| \leq \left\|f(x+y+z)\right\|$$

in Banach spaces. In 2011, J. R. Lee, C. Park and D. Y. Shin [2] studied the generalized Hyers-Ulam stability of the additive functional inequality

$$\left\|f(2x) + f(2y) + 2f(z)\right\| \leq \left\|2f(x+y+z)\right\|$$

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in Banach spaces.

In this paper, we give some generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(2x_1)+f(2x_2)+\cdots+f(2x_n)\| \le \|tf(x_1+x_2+\cdots+x_n)\|$$

in Banach spaces where $3 \le n$ and $2 \le t < n (n \in \mathbb{Z}$ and $t \in \mathbb{R})$.

2. Hyers-Ulam stability in Banach spaces

To obtain our main result, we need the following lemma.

LEMMA 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping and let $3 \le n$ and $2 \le t < n$ where n is an integer and t is a real number. Then f is additive if and only if it satisfies*

$$\|f(2x_1)+f(2x_2)+\cdots+f(2x_n)\| \le \|tf(x_1+x_2+\cdots+x_n)\| \tag{2.1}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$.

Proof. If f is additive, then clearly

$$\big\|f(2x_1)+\cdots+f(2x_n)\| = \|2f(x_1+\cdots+x_n)\| \le \|tf(x_1+\cdots+x_n)\|$$

for all $x_i \in \mathcal{X}$.

Conversely, assume that f satisfies (2.1). Letting $x_i = 0$ in (2.1), we gain $\big\|nf(0)\big\| \le \big\|tf(0)\big\|$ and so $f(0) = 0$ by the assumpution. Putting $x_i = 0$ for all $i = 3, \cdots, n$, and replacing x_1, x_2 by $x, -x$ in (2.1), we get $\big\|f(-x)+f(x)\big\| \le \big\|tf(0)\big\| = 0$ and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$. Setting $x_1 = \dfrac{x+y}{2}, x_2 = \dfrac{-x}{2}, x_3 = \dfrac{-y}{2}, x_i = 0 (4 \le i \le n)$ in (2.1), we have

$$\big\|f(x+y)+f(-x)+f(-y)\big\| \le \big\|tf(0)\big\| = 0$$

for all $x, y \in \mathcal{X}$. Thus we obtain $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathcal{X}$. □

THEOREM 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$ and let $3 \le n$ and $2 \le t < n$. If there is a function $\varphi : \mathcal{X}^n \to [0, \infty)$ satisfying*

$$\|f(2x_1)+\cdots+f(2x_n)\| \le \|tf(x_1+\cdots+x_n)\| + \varphi(x_1, \cdots, x_n) \tag{2.2}$$

and

(2.3)

$$\widetilde{\varphi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\big((-2)^j x_1, (-2)^j x_2, (-2)^j x_3, x_4, \cdots, x_n\big) < \infty$$

for all $x_1, \cdots, x_n \in \mathcal{X}$*, then there exists a unique additive mapping* $A : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - A(x)\| \le \frac{1}{2}\widetilde{\varphi}\left(x, -\frac{x}{2}, -\frac{x}{2}, 0, \cdots, 0\right) \tag{2.4}$$

for all $x \in \mathcal{X}$.

Proof. Replacing $x_1, x_2, x_3, x_i (4 \le i)$ by $(-2)^{n+1}\frac{x}{2}, (-2)^n\frac{x}{2}, (-2)^n\frac{x}{2}, 0,$ respectively, and dividing by 2^{n+1} in (2.2), since $f(0) = 0$, we get

$$\begin{aligned}&\left\|\frac{f\big((-2)^{n+1}x\big)}{(-2)^{n+1}} - \frac{f\big((-2)^n x\big)}{(-2)^n}\right\| \\ &\le \frac{1}{2^{n+1}}\varphi\left((-2)^{n+1}\frac{x}{2}, (-2)^n\frac{x}{2}, (-2)^n\frac{x}{2}, 0, \cdots, 0\right)\end{aligned}$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. From the above inequality, we have

$$\begin{aligned}&\left\|\frac{f\big((-2)^n x\big)}{(-2)^n} - \frac{f\big((-2)^m x\big)}{(-2)^m}\right\| \\ &\le \sum_{j=m}^{n-1}\left\|\frac{f\big((-2)^{j+1}x\big)}{(-2)^{j+1}} - \frac{f\big((-2)^j x\big)}{(-2)^j}\right\| \\ &\le \sum_{j=m}^{n-1}\frac{1}{2^{j+1}}\varphi\big((-2)^j x, (-2)^{j-1}x, (-2)^{j-1}x, 0, \cdots, 0\big)\end{aligned} \tag{2.5}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. By the condition (2.3), the sequence $\left\{\frac{f((-2)^n x)}{(-2)^n}\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{\frac{f((-2)^n x)}{(-2)^n}\right\}$ converges for all $x \in \mathcal{X}$. So we can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty}\frac{f\left((-2)^n x\right)}{(-2)^n}$$

for all $x \in \mathcal{X}$.

In order to prove that A satisfies (2.4), taking $m = 0$ and letting n tend to ∞ in (2.5), then we have the following inequality (2.4).

$$\begin{aligned}\Big\|A(x) - f(x)\Big\| &\le \sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\varphi\big((-2)^j x, (-2)^{j-1}x, (-2)^{j-1}x, 0, \cdots, 0\big) \\ &= \frac{1}{2}\widetilde{\varphi}\left(x, -\frac{x}{2}, -\frac{x}{2}, 0, \cdots, 0\right).\end{aligned}$$

Next we show that A is additive. Replacing x_i by $(-2)^n x_i$ for all $i = 1, 2, \cdots, n$, and dividing by 2^n in (2.2), we obtain

$$\begin{aligned}
&\left\| \frac{f\big((-2)^n 2x_1\big)}{(-2)^n} + \frac{f\big((-2)^n 2x_2\big)}{(-2)^n} + \cdots + \frac{f\big((-2)^n 2x_n\big)}{(-2)^n} \right\| \\
\leq\ & \left\| t \frac{f\left((-2)^n (x_1 + x_2 + \cdots + x_n)\right)}{(-2)^n} \right\| \\
&+ \frac{1}{2^n} \varphi\big((-2)^n x_1, (-2)^n x_2, \cdots, (-2)^n x_n\big)
\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$ and all nonnegative integers n. Since (2.3) gives that

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi\big((-2)^n x_1, (-2)^n x_2, \cdots, (-2)^n x_n\big) = 0$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$, letting n tend to ∞ in the above inequality, we have

$$\left\| A\big(2x_1\big) + A\big(2x_2\big) + \cdots + A\big(2x_n\big) \right\| \leq \left\| tA\big(x_1 + x_2 + \cdots + x_n\big) \right\|$$

so A is additive by Lemma 2.1.

Let $A' : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (2.4). Since both A and A' are additive, we have, for all positive integer n

$$\begin{aligned}
&\big\| A(x) - A'(x) \big\| \\
=\ & \frac{1}{2^n} \big\| A\big((-2)^n x\big) - A'\big((-2)^n x\big) \big\| \\
\leq\ & \frac{1}{2^n} \left(\big\| A\big((-2)^n x\big) - f\big((-2)^n x\big) \big\| + \big\| f\big((-2)^n x\big) - A'\big((-2)^n x\big) \big\| \right) \\
\leq\ & \frac{1}{2^n} \widetilde{\varphi}\big((-2)^n x, (-2)^{n-1} x, (-2)^{n-1} x, 0, \cdots, 0\big) \\
=\ & \sum_{j=n}^{\infty} \frac{1}{2^j} \varphi\big((-2)^{j-n} x, (-2)^{j-1-n} x, (-2)^{j-1-n} x, 0, \cdots, 0\big)
\end{aligned}$$

which goes to zero as $n \to \infty$ for all $x \in \mathcal{X}$ by (2.3). Therefore, A is a unique additive mapping satisfying (2.4), as desired. □

THEOREM 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping and let $3 \leq n$ and $2 \leq t < n$. If there is a function $\varphi : \mathcal{X}^n \to [0, \infty)$ satisfying*

$$\|f(2x_1) + \cdots + f(2x_n)\| \leq \|tf(x_1 + \cdots + x_n)\| + \varphi(x_1, \cdots, x_n). \tag{2.6}$$

where

$$\begin{aligned}(2.7)\qquad &\widetilde{\varphi}(x_1, x_2, \cdots, x_n)\\ &:= \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x_1}{(-2)^j}, \frac{x_2}{(-2)^j}, \frac{x_3}{(-2)^j}, x_4, \cdots, x_n\right) < \infty\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$, *then there exists a unique additive mapping* $A : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - A(x)\| \leq \frac{1}{2}\widetilde{\varphi}\left(x, -\frac{x}{2}, -\frac{x}{2}, 0, \cdots, 0\right) \tag{2.8}$$

for all $x \in \mathcal{X}$.

Proof. We have $\varphi(0, \cdots, 0) = 0$ by (2.7), and so $f(0) = 0$ by (2.6). Replacing $x_1, x_2, x_3, x_i (4 \leq i)$ by $\frac{x}{(-2)^n}, \frac{x}{(-2)^{n+1}}, \frac{x}{(-2)^{n+1}}, 0$, respectively, and multiplying by 2^{n-1} in (2.6), since $f(0) = 0$, we get

$$\begin{aligned}&\left\|(-2)^{n-1} f\left(\frac{x}{(-2)^{n-1}}\right) - (-2)^n f\left(\frac{x}{(-2)^n}\right)\right\|\\ &\leq \ 2^{n-1}\varphi\left(\frac{x}{(-2)^n}, \frac{x}{(-2)^{n+1}}, \frac{x}{(-2)^{n+1}}, 0, \cdots, 0\right)\end{aligned}$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. From the above inequality, we have

$$\begin{aligned}(2.9)\qquad &\left\|(-2)^n f\left(\frac{x}{(-2)^n}\right) - (-2)^m f\left(\frac{x}{(-2)^m}\right)\right\|\\ &\leq \sum_{j=m+1}^{n} \left\|(-2)^j f\left(\frac{x}{(-2)^j}\right) - (-2)^{j-1} f\left(\frac{x}{(-2)^{j-1}}\right)\right\|\\ &\leq \sum_{j=m+1}^{n} 2^{j-1}\varphi\left(\frac{x}{(-2)^j}, \frac{x}{(-2)^{j+1}}, \frac{x}{(-2)^{j+1}}, 0, \cdots, 0\right)\end{aligned}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. By the condition (2.7), the sequence $\left\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\}$ converges for all $x \in \mathcal{X}$. So we can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} \left\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\}$$

for all $x \in \mathcal{X}$.

In order to prove that A satisfies (2.8), taking $m = 0$ and letting n tend to ∞ in (2.9), then we have the following inequality (2.8).

$$\begin{aligned}\Big\|A(x) - f(x)\Big\| &\le \sum_{j=0}^{\infty} 2^{j-1}\varphi\left(\frac{x}{(-2)^j}, \frac{x}{(-2)^{j+1}}, \frac{x}{(-2)^{j+1}}, 0, \cdots, 0\right) \\ &= \frac{1}{2}\widetilde{\varphi}\left(x, -\frac{x}{2}, -\frac{x}{2}, 0, \cdots, 0\right).\end{aligned}$$

Next we show that A is additive. Replacing x_i by $\dfrac{x_i}{(-2)^n}$ for all $i = 1, 2, \cdots, n$, and multiplying by 2^n in (2.6), we obtain

$$\begin{aligned}&\left\|(-2)^n f\left(\frac{2x_1}{(-2)^n}\right) + (-2)^n f\left(\frac{2x_2}{(-2)^n}\right) + \cdots + (-2)^n f\left(\frac{2x_n}{(-2)^n}\right)\right\| \\ &\le \left\|t(-2)^n f\left(\frac{(x_1 + x_2 + \cdots + x_n)}{(-2)^n}\right)\right\| \\ &\quad + 2^n\varphi\left(\frac{x_1}{(-2)^n}, \frac{x_2}{(-2)^n}, \cdots, \frac{x_n}{(-2)^n}\right)\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$ and all nonnegative integers n. Since (2.7) gives that

$$\lim_{n\to\infty} 2^n\varphi\left(\frac{x_1}{(-2)^n}, \frac{x_2}{(-2)^n}, \cdots, \frac{x_n}{(-2)^n}\right) = 0$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$, letting n tend to ∞ in the above inequality, we have

$$\Big\|A(2x_1) + A(2x_2) + \cdots + A(2x_n)\Big\| \le \Big\|tA(x_1 + x_2 + \cdots + x_n)\Big\|$$

so A is additive by Lemma 2.1.

Let $A' : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (2.8). Since both A and A' are additive, we have, for all positive integer n

$$\begin{aligned}&\|A(x) - A'(x)\| \\ &= 2^n\left\|A\left(\frac{x}{(-2)^n}\right) - A'\left(\frac{x}{(-2)^n}\right)\right\| \\ &\le 2^n\left(\left\|A\left(\frac{x}{(-2)^n}\right) - f\left(\frac{x}{(-2)^n}\right)\right\| + \left\|f\left(\frac{x}{(-2)^n}\right) - A'\left(\frac{x}{(-2)^n}\right)\right\|\right) \\ &\le 2^n\widetilde{\varphi}\left(\frac{x_1}{(-2)^n}, \frac{-x_2}{(-2)^{n+1}}, \frac{-x_3}{(-2)^{n+1}}, 0, \cdots, 0\right)\end{aligned}$$

$$= \sum_{j=n+1}^{\infty} 2^j \varphi\left(\frac{x_1}{(-2)^{j-n}}, \frac{-x_2}{(-2)^{j+1-n}}, \frac{-x_3}{(-2)^{j+1-n}}, 0, \cdots, 0\right)$$

which goes to zero as $n \to \infty$ for all $x \in \mathcal{X}$ by (2.7). Therefore, A is a unique additive mapping satisfying (2.8), as desired. $\square$

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